A Numerical Method for the Solution of the Junction of Cylindrical Waveguides

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Abstract—**In order to solve the waveguide junction problem numerically, we express the fields in the guides by truncated modal expansions and construct an error function which is a measure of the mean-square error in the matching of the boundary conditions at the junction. The minimum of this error leads to a set of linear equations for the modal amplitudes. Offset rectangular waveguides with amplitude and current excitations are studied. A weighting factor which multiplies the error contribution due to the magnetic field is studied, and a criterion for its selection given.**

Introduction

References [1]–[4] propose essentially the same numerical technique for the solution of various waveguide discontinuities. Their approach basically consists of expressing the fields in different sections of a waveguide by their truncated modal expansions. A set of linear equations is then obtained for the unknown modal coefficients by cross multiplying the boundary condition expressions at the discontinuities by the appropriate mode functions and integrating over the boundary surface using the orthogonality conditions among the modes. These methods exhibit a phenomenon of relative convergence first discussed in [1]. It is also shown in [4] that for thin iris discontinuity, the aperture field modes should be less than the guide modes, otherwise the method fails. These methods have to treat the boundary reduction and enlargement separately. The method described here does not suffer from these shortcomings, leading to very stable matrices, since the main diagonal elements have usually the largest magnitudes.

Davies has recently written on the mode-matching technique by a least-square criterion [5], which parallels our approach. However, the research reported here has been independent of his study and has led, we believe, to a simpler procedure for computer implementation.

Theoretical Development

Consider the junction of two air-filled cylindrical waveguides as shown in Fig. 1, and assume, for simplicity, that the second guide is matched. Note that this is not a requirement imposed by the technique. We express the fields as

$$\begin{aligned} E^1 &= \sum_m (A_m^+ \exp(-\gamma_m z) + A_m^- \exp(\gamma_m z))e_m^1, \qquad z<0\\ H^1 &= \sum_m (A_m^+ \exp(-\gamma_m z) - A_m^- \exp(\gamma_m z))h_m^1\\ E^2 &= \sum_p B_p \exp(-\Gamma_p z)e_p^2\\ H^2 &= \sum_p B_p \exp(-\Gamma_p z)e_p^2, \qquad z>0. \end{aligned} \tag{1}$$

The guide is excited by an incident mode with amplitude A_m^+ and/or an electric current sheet J and/or a magnetic current sheet M over some portion of the junction surface.

To match the boundary conditions (BC) a so-called error function (ϵ) is constructed by summing the products of each tangential BC with its complex conjugate and integrating over the boundary surface.

$$\epsilon = \alpha \int_{ap} | u_z x(H^2 - H^1) - J|^2 ds + \int_{ap} | u_z \times (E^2 - E^1) + M|^2 ds + \int_{c(1)} |E^1|^2 ds + \int_{c(2)} |E^2|^2 ds \tag{2}$$

where α is a weighting factor to balance the contribution of the magnetic field to the error, $c(1)$ and $c(2)$ represent the metallic diaphragm surfaces at the junction toward the first and second guides, respectively, and ap denotes the aperture surface. (The BC on the normal field components are contained in those of the tangential ones.)

The error becomes zero whenever the BC's are perfectly matched. Due to the uniqueness of the fields, the unique minimum of ϵ gives the set of modes generated by the discontinuity. For a finite number of modes, the minimum gives the best matching of the BC's in the mean-square sense. In contrast, other methods apparently do not have such a physical criterion.

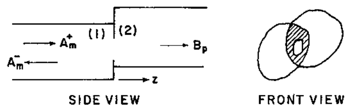

Fig. 1. Junction of two cylindrical waveguides.

The minimum of ϵ can be obtained by any minimization scheme. However, we chose the following procedure. The partial derivatives of ϵ with respect to the real and imaginary parts of the modal amplitudes vanish at its minimum; equivalently, the partial derivatives with respect to the modal amplitudes and their complex conjugates vanish at the minimum [7]. Since ϵ is real, we have $(\partial\epsilon/\partial A_m^-) = (\partial\epsilon/\partial A_m^{-*}).^*$ (Asterisk denotes complex conjugate.) Consequently, ϵ may be minimized by equating to zero its partial derivatives with respect to the conjugate of the modal amplitudes only.

We may obtain this set of linear equations directly from the boundary condition expressions without actually constructing ϵ in (2) by the argument outlined below. The tangential boundary conditions may be written as

$$\begin{bmatrix} \sqrt{\alpha}\, u_z\times h_m^1 & \sqrt{\alpha}\, u_z\times h_p^2 \\ -e_m^1\times u_z & e_p^2\times u_z \\ -e_m^1\times u_z & 0 \\ 0 & e_p^2\times u_z \end{bmatrix} \begin{bmatrix} A_m^- \\ \\ B_p \\ \end{bmatrix} = \begin{bmatrix} \sqrt{\alpha} J + \sqrt{\alpha}\sum_m A_m^+ u_z\times h_m^1 \\ M + \sum_m A_m^+ e_m^1\times u_z \\ \sum_m A_m^+ e_m^1\times u_z \\ 0 \end{bmatrix} \tag{3}$$

where $m=1, 2, \cdots, N$ and $p=1, 2, \cdots, P$ are column indices. The first and second rows are due to the BC's on the H and E fields; the third and fourth rows are due to the vanishing of the tangential E over the diaphragm toward the left and right sides of the junction. If there were any magnetic conductors at the discontinuity, the vanishing of the tangential H field should be also included.

Assuming a finite number of modes, the partial derivative of the error due to the H field (ϵ^h) with respect to A_n^{-*} is obtained by scalar multiplication of the BC on the H field by $\sqrt{\alpha}\, u_z\times h_n^{1*}$, the conjugate of the coefficient of A_n^- in it, and integrating each term wherever valid. Similarly for the other terms in $(\partial\epsilon/\partial A_n^{-*})$. We then see that $(\partial\epsilon/\partial A_n^{-*})$ may be obtained by scalar premultiplication of (3) by the conjugate transpose of the nth column of the coefficient matrix in (3) corresponding to A_n^{-*}, and integrating. Similarly for $(\partial\epsilon/\partial B_q^{-*})$. Therefore, the partial derivatives of ϵ can be simply obtained by scalar premultiplication of (3) by the conjugate transpose of the coefficient matrix, and integrating wherever valid over the junction boundary surface.

Equation (3) may be written as

$$LV = f \tag{4}$$

where L is the coefficient matrix, $V' = (A^-, B)$, and f is the forcing function in (3). The above operations may be concisely denoted by

$$\begin{aligned} \langle L^*, L\rangle V &= \langle L^*, f\rangle\\ V &= \langle L^*, L\rangle^{-1}\langle L^*, f\rangle. \end{aligned} \tag{5}$$

Since the coefficient matrix $\langle L^*, L\rangle$ is Hermitian, it is only necessary to compute and store its triangular portion. Faster routines are also available for the inversion of Hermitian matrices [8]. The diagonal elements of the coefficient matrix are usually larger than the magnitude of the off-diagonal elements which leads to a stable matrix inversion. The cases of the boundary reduction and enlargement are both contained in the general formulation, and it has not been necessary to introduce an aperture field. The number of modes in the two guides also can be selected independently, and their ratio is unimportant since ϵ always has a minimum.

In this notation, the error is

$$\epsilon = V^*\langle L^*, L\rangle V - V^*\langle L^*, f\rangle - \langle f^*, L\rangle V + \langle f^*, f\rangle.$$

Although the method is developed for the junction of two cylindrical waveguides, it is equally applicable to any type of discontinuity. The procedure is to write all the boundary conditions over the interfaces between different regions in a matrix equation as in (4) and match the BC's by performing the operations denoted by (5).

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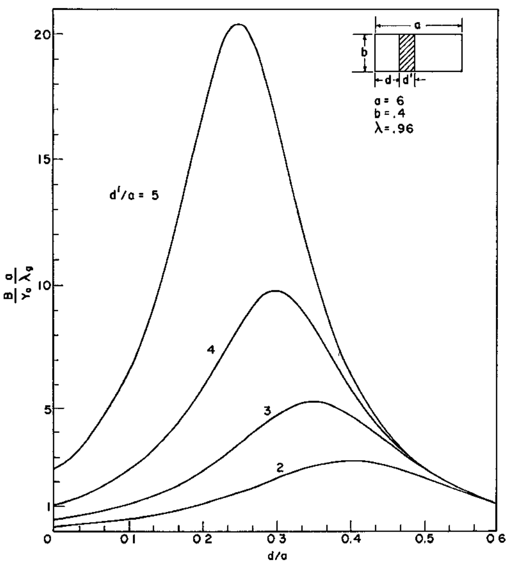

Fig. 2. Normalized susceptance of an inductive metallic strip—family of curves.

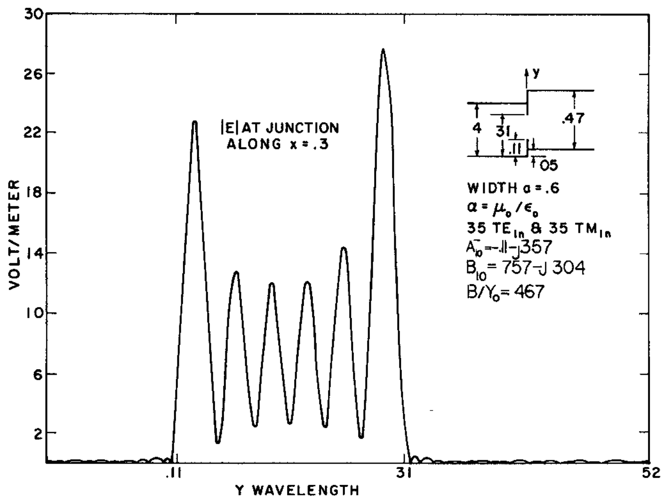

Fig. 3. Capacitive junction of two rectangular waveguides—$|E|$ at junction.

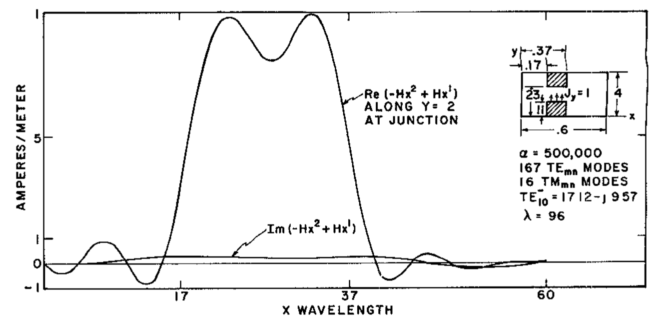

Fig. 4. Current probe excitation of a rectangular waveguide.

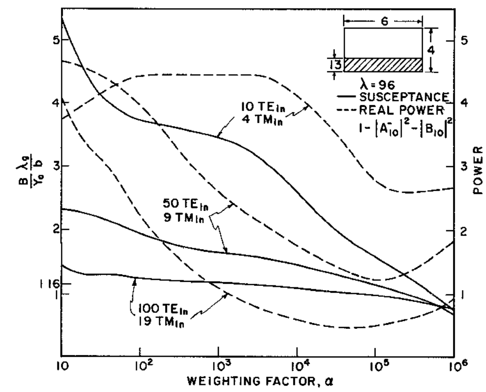

Fig. 5. Susceptance and real power—asymmetric capacitive metallic strip.

Numerical Results

Computer programs are available for the junction of two offset rectangular waveguides. The guides may be displaced with respect to each other, and may have different dimensions. However, their axes are assumed parallel. There may be amplitude excitation (TE or TM) and current excitation (constant current sheet over some aperture). The junction may have any number of apertures.

It is necessary to assume a general field, as the sum of TE and TM modes, inside the rectangular waveguide. The types of modes that a discontinuity generates for some excitation may be deduced by an appropriate arrangement of the elements of (5). For example, we may show that for dominant TE_{10}, mode excitation, the inductive discontinuity generates TE_{m0} modes only, and the capacitive discontinuity generates TE_{1n} and TM_{1n} modes. The symmetrical inductive diaphragm generates odd TE_{m0} modes, etc. [10].

Our computer programs can handle many cases not treated in the literature. We reproduce here some typical examples. In Fig. 2 we plot the susceptance $(B/Y_0)(a/\lambda_g)$ of an inductive metallic strip as a function of its position d/a over the cross section of the rectangular waveguide with its width d'/a as the parameter. The symmetric case when the strip is in the middle and the asymmetric case when it is located at one corner of the broadside correspond with those obtained from *Waveguide Handbook* [9]. Fig. 3 plots the magnitude of the electric field along $x=a/2=0.3$ for the capacitive junction of two offset rectangular waveguides. It oscillates over the aperture, increases sharply at the edges, and tends to zero over the metallic diaphragms as it expectedly does in the related similar junction of parallel plate waveguides. In Fig. 4 we plot $(H_x^2-H_x^1)$ along $y=b/2=0.2$ for a waveguide excited by a constant y-directed current probe across the gap of a zero thickness metallic post. The discontinuity in H_x at the junction should be equal to the y-directed impressed current, which is constant inside the gap and zero everywhere else. The programs can handle many other cases [10].

The offset junction of two parallel plate waveguides has also been formulated. To check the method, we have studied the case of step-down discontinuity in detail. The field components satisfy the boundary conditions, as well as the edge condition ($|A_n^-| \sim n^{-n/3}$ for large n) [10].

Weighting Factor

The weighting factor α provides a degree of freedom in the sense that its proper choice may give good values for the equivalent susceptance with relatively small number of modes. Let

$$\epsilon = \epsilon^e + \alpha\epsilon^h$$

where ϵ^e and ϵ^h denote contributions to the error due to the E and H fields, respectively.

We now state some properties of ϵ whose proofs may be found in [10]. The minimum value of ϵ ($\epsilon_{\min}$) is a monotone increasing function of α. Therefore, $\epsilon_{\min}$ cannot be used as a criterion for the selection of α. Increasing α tends to better the satisfaction of the BC on the H field (ϵ^h decreases) and worsen that due to the E field (ϵ^e increases) and vice versa. $\epsilon_{\min}$ is a decreasing function of the mode numbers.

The conservation of real power may serve as the criterion for the selection of α at least as far as the equivalent susceptance is concerned, since both are computed from the propagating modal amplitudes [6, p. 19]. The discrepancy in the conservation of real power (P_r) for an asymmetric capacitive strip and its equivalent susceptance are plotted in Fig. 5. For larger number of modes the susceptance is less sensitive to the variation of α as indicated by smaller slope of its curve. P_r also progressively decreases for higher number of modes. It has a a broad minimum against α, and for the corresponding values of α the susceptance changes only slightly. We may thus obtain the range of appropriate values of α. The correct value of the normalized susceptance is 1.16 and occurs at about the minimum of P_r.

It is desirable to obtain an estimate of α for the most accurate equivalent susceptance, or derive an expression for V in (5) such that the modal amplitudes may be readily obtained without having to invert $\langle L^*, L\rangle$ for each α.

The matrices in (5) can be written with the α dependence made explicit in the following way:

$$(M_1 + \alpha M_2)V = (f_1 + \alpha f_2). \quad (6)$$

Since M_1 and M_2 are Hermitian, the weighted eigenvalue equation

$$M_2 v_i = \lambda_i M_1 v_i$$

has real eigenvalues, and its eigenfunctions are orthogonal with respect to the weights M_1 and M_2. Then, substituting

$$V = \sum \beta_i v_i$$

into (6) and multiplying the resulting equation by v_j^* to obtain β_j, we get

$$V = \sum_i \frac{v_i^*(f_1 + \alpha f_2)}{1 + \alpha\lambda_i} v_i$$

assuming normalized eigenvectors (divide v_i by $(v_i^* M_1 v_i)^{1/2}$). A similar expression can be obtained by expanding f_1 and f_2 [10]. Once λ_i and v_i are computed we may obtain V for different values of α. We may also obtain an estimate for α by substituting the propagating modal amplitudes into the expression of the conservation of real power and imposing approximations $|\alpha\lambda_i| \ll 1$ and/or $|\alpha\lambda_i| \gg 1$. Other expressions for V are given in [10].

Conclusion

A numerical method for the solution of waveguide discontinuities has been developed here which is suitable for computer implementation and which does not suffer from some of the shortcomings of the other methods. We have solved many problems numerically which do not appear in the extant literature. The method can as well handle other types of waveguides and discontinuities.

The problem that remains to be solved is that of an easier criterion for the selection of the weighting factor (α) so that the smallest possible number of modes can be used for a given accuracy. Davies [5] uses one among several condition numbers of the matrices as the criterion for the selection of α. Such a condition number may be an indicator of the stability of the matrix inversion, but its relation to the equivalent susceptance of the discontinuities and the dominant modal amplitudes remains obscure.

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The Effect of Surface Metal Adhesive on Slot-Line Wavelength

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Abstract—**An investigation of the dependence of slot-line wavelength upon a thin layer of adhesive between metal and substrate is described. It is shown that the presence of adhesive will cause an increase in wavelength when the dielectric constant of the adhesive is less than that of the substrate. Experimental results are presented which show this dependence for a variety of surfaces and adhesives. A perturbation expression is given which permits correction of experimental data for comparison with theory when this effect occurs.**

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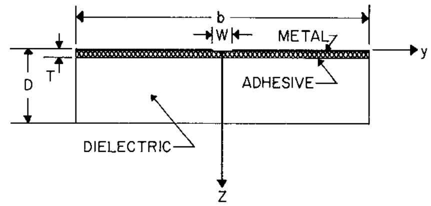

Fig. 1. Slot-line geometry with adhesive.

TABLE I

Thickness of Metals and Adhesives

Metallization	Thickness of Metal (Mils)	Thickness of Adhesive (Mils)
Factory 1 Oz. Copper	1.15	< 1.0
3M Copper Tape	1.25	1.9
3M Aluminum Tape	1.95	1.8
Circuit-Stik Copper Foil	1.15	2.3
Evaporated Copper	0.65	0

I. Introduction

The analysis of slot line and its microwave applications have been discussed by a number of authors [1]–[7] during the past several years. In one of these papers, Mariani *et al.* [5] presented measured values of slot wavelength for various substrates metallized with both aluminum sensing tape and copper (electroless plated). Their data showed that the slot wavelength on substrates metallized with aluminum sensing tape exceeded the theoretical value. For substrates with copper plated surfaces, the measured wavelength was (with one exception) somewhat less than the theoretical wavelength. It was concluded that the adhesive which was present in the case of aluminum sensing tape decreased the effective dielectric constant and thereby increased slot wavelength.

Measurements in our laboratory substantiate this conclusion. The purpose of this short paper is to present more consistent and extensive data on this effect and to treat the problem using perturbation theory.

II. Slot-Wavelength Measurements

Slot line is constructed by etching a slot utilizing a dielectric substrate which has been metallized on one side only. The metal may be applied in various ways and in some cases a thin layer of adhesive is present between the metal and the substrate as illustrated in Fig. 1. This adhesive may have a significant effect upon the slot wavelength.

A number of experiments were conducted to investigate adhesive effects. In one series of experiments a Custom Materials Hi-K707-20 ($\epsilon_r = 20$) substrate was tested using several different methods of metallization. The substrate was 3-in wide by 0.125-in thick, and slot width was maintained constant in all cases with $W/D = 0.53 \pm 0.02$. The surfaces tested were 1-oz copper as supplied by the manufacturer, 3M copper tape (1-in wide), 3M aluminum tape (1-in wide), and a vacuum deposited copper surface. Table I lists the thicknesses of metal and adhesive for all surfaces tested.

Measured values of λ'/λ for these surfaces are displayed in Fig. 2 along with the theoretical curve from [5]. The vacuum deposited copper surface is in intimate contact with the substrate and the wavelength ratio for this surface may be used as a basis for comparison of experimental measurements. All other surfaces are separated from the substrate by an adhesive layer and increased wavelength ratios result.